Shock-Capturing Method for the Equations of Gasdynamics in Physical Variables

Ijaz H. Parpia*
University of Texas at Arlington, Arlington, Texas 76019

Abstract

An upwind shock-capturing scheme is proposed for the equations of gasdynamics in nonconservative form. The new method is constructed so that stationary discontinuities aligned with the computational mesh are resolved perfectly. The method fails to be strictly conservative across smeared discontinuities, but we show by numerical example that the conservation errors have only a small effect on the overall strength and speed of propagation of smeared waves.

Contents

A number of very successful methods for the solution of the equations of gasdynamics are based on the theory of characteristics: The wave nature of the problem emerges very clearly when the governing equations are written in characteristic form. Characteristics-based schemes have been developed for both the conservative and the nonconservative forms of the governing equations.[1-5] Of particular interest here are nonconservative methods. We shall outline one of these, the Split Coefficient Matrix (SCM) scheme,[5] which is related to the shock-capturing algorithm we develop here. Consider the governing equations for one-dimensional flow in nonconservative form

$$\frac{\partial v}{\partial t} + A\,\frac{\partial v}{\partial x} = 0 \tag{1}$$

where $v = (\rho,\ u,\ p)$ (density, particle velocity, and pressure, respectively) is the vector of dependent variable. The system of Eq. (1) is hyperbolic, therefore the matrix A can be diagonalized by the similarity transformation

$$\Lambda = T^{-1}AT$$

where Λ is the diagonal matrix of (real) eigenvalues of A, and T is the matrix of right eigenvectors. The matrix of eigenvalues can be split into positive and negative parts according to

$$\Lambda^{\pm} = \text{diag}\,(\lambda_i^{\pm});\ \lambda_i^{\pm} = \frac{\lambda_i \pm |\lambda_i|}{2}$$

so that A can be written as the sum

$$A = T(\Lambda^+ + \Lambda^-)T^{-1} = A^+ + A^-$$

and the term $A\,\partial v/\partial x$ can be understood to represent the contributions of weak forward- and backward-traveling waves with speeds λ_i^+ and λ_i^-, respectively.

This procedure defines the SCM scheme, which for the system of Eq. (1) can be written

$$\Delta^{n+\frac{1}{2}} V = -\nu[A_i^+\,\Delta_{i-\frac{1}{2}} V + A_i^-\,\Delta_{i+\frac{1}{2}} V] \tag{2}$$

where $\nu = \Delta t/\Delta x$, and $\Delta^{n+\frac{1}{2}}(\ ) = (\ )_i^{n+1} - (\ )_i^n$. The SCM scheme is extended to the case of multidimensional flow in the usual way, using operator splitting.[1] A generalization of the SCM scheme for Rankine-Hugoniot (RH) waves of arbitrary strength can be achieved by reconstructing each jump using a superposition of eigensolutions of the RH conditions

$$\bar{A}\Delta V = U\Delta V \tag{3}$$

where $\Delta V = V_r - V_l$. The $\bar{A}(V_l, V_r)$ involves simple arithmetic averages of the flow properties V_l and V_r.[6] Moreover, $\bar{A}$ has real eigenvalues and a full set of linearly independent eigenvectors and can therefore be diagonalized:

$$\bar{\Lambda} = \bar{T}^{-1}\bar{A}\bar{T} \tag{4}$$

and subsequently split following the procedure used earlier in forming $A^{\pm}$. It therefore follows that the discretization

$$\Delta^{n+\frac{1}{2}} V = -\nu[\bar{A}^+_{i-\frac{1}{2}}\Delta_{i-\frac{1}{2}} V + \bar{A}^-_{i+\frac{1}{2}}\Delta_{i+\frac{1}{2}} V] \tag{5}$$

in which $\bar{A}^{\pm}_{i\pm\frac{1}{2}}$ are formed from V_i, $V_{i\pm1}$, is exactly conservative in steady one-dimensional flow, because stationary discontinuities are perfectly resolved. For moving discontinuities, and in multidimensional problems on arbitrary meshes, the algorithm (and its simple extension to multidimensional problems) fails to be conservative. The errors incurred are, however, generally very small in transonic flow.

It is possible to apply a correction to the algorithm that ensures that smeared strong discontinuities are more accurately modeled. The development of the modified algorithm is presented here in the general framework of flux-difference splitting. For one-dimensional flow, the conservative form of the governing equations is

$$\frac{\partial \boldsymbol{q}}{\partial t} + \frac{\partial \mathfrak{F}}{\partial x} = 0 \tag{6}$$

where $\boldsymbol{q} = (\rho, \rho u, \rho e)$ is the vector of conserved variables, and $\mathfrak{F}$ is the flux vector. A discrete approximation to Eq. (6) is conservative provided that it can be written as

$$Q_i^{n+1} = Q_i^n - \nu(F_{i+\frac{1}{2}} - F_{i=\frac{1}{2}}) \tag{7}$$

where Q_i^n is an approximation to $\boldsymbol{q}(i\Delta x, n\Delta t)$, and $F_{i+\frac{1}{2}} = F_{i+\frac{1}{2}}(Q_{i-q+1}, \ldots, Q_{i+q})$ is a consistent approximation to the physical flux at the cell interface between the grid points i and $i+1$.

The flux difference between neighboring states $\boldsymbol{q}_l$ and $\boldsymbol{q}_r$ can always be written as

$$\mathfrak{F}_r - \mathfrak{F}_l = \int_{q_l}^{q_r} B\,\mathrm{d}\boldsymbol{q} \tag{8}$$

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*Assistant Professor, Department of Aerospace Engineering. Member AIAA.

where $B = \partial \mathfrak{F}/\partial q$. Roe[2] defines a matrix $\tilde{B}(q_l, q_r)$ such that for any q_l and q_r,

$$\mathfrak{F}_r - \mathfrak{F}_l = \tilde{B}(q_r - q_l) \tag{9}$$

and furthermore, wherever q_l and q_r are separated by an RH discontinuity, moving with speed U,

$$\tilde{B}(q_r - q_l) = U(q_r - q_l) \tag{10}$$

For jumps Δq, Δv, we have

$$\Delta q = \bar{M}\Delta v \tag{11}$$

where $\bar{M}$ is the arithmetically averaged Jacobian $\partial q/\partial v$. Using Eq. (3) together with Eq. (11), it follows that

$$\tilde{B}\Delta q = \bar{M}\bar{A}\bar{M}^{-1}\Delta q = \bar{B}\Delta q$$

for jumps Δq and Δv across and RH discontinuity. We note that, in general,

$$\Delta\mathfrak{F} \neq \bar{B}\Delta q$$

Evidently, a flux difference splitting scheme in which $\bar{B}$ is used in place of $\tilde{B}$ is not conservative; we shall show by numerical example that the errors we accept in using $\bar{B}$ are small.

The matrix $\bar{B}$ is similar to $\bar{A}$, so that we can define a splitting

$$\bar{B} = \bar{B}^+ + \bar{B}^- = \bar{M}(\bar{A}^+ + \bar{A}^-)\bar{M}^{-1}$$

We are now able to define the discrete form of the proposed scheme:

$$\Delta^{n+\frac{1}{2}}Q = -\nu(\bar{B}^-_{i+\frac{1}{2}}\Delta_{i+\frac{1}{2}}Q + \bar{B}^+_{i-\frac{1}{2}}\Delta_{i-\frac{1}{2}}Q) \tag{12}$$

or, in terms of the physical variables,

$$\bar{M}_{n+\frac{1}{2}}\Delta^{n+\frac{1}{2}}V = -\nu(\bar{M}_{i+\frac{1}{2}}\bar{A}^-_{i+\frac{1}{2}}\Delta_{i+\frac{1}{2}}V + \bar{M}_{i-\frac{1}{2}}\bar{A}^+_{i-\frac{1}{2}}\Delta_{i-\frac{1}{2}}V) \tag{13}$$

Equation (13) is readily extended to multidimensional problems by operator splitting.

Denoting the spatial-difference terms by the vector $r = (r^{(1)}, r^{(2)}, r^{(3)})^T$, we have, in one-dimensional problems,

$$\rho^{n+1} = \rho^n - r^{(1)}\Delta t$$

$$u^{n+1} = (\rho^n u^n - r^{(2)}\Delta t)/\rho^{n+1}$$

$$p^{n+1} = p^n - (\gamma - 1)\left[r^{(3)}\Delta t + \tfrac{1}{2}\left(\rho^{n+1}(u^{n+1})^2 - \rho^n(u^n)^2\right)\right]$$

To illustrate the performance of the method, two test cases are considered. In the first example, the initial conditions were set corresponding to a shock wave moving from left to right at a Mach number of 10. The density, pressure, and velocity distributions after 40 time steps, with $\Delta t/\Delta x = 0.075$, are compared to the exact solution, and to a solution using Roe's scheme, in Fig. 1. The method was also used to calculate the flow over a cylinder in a freestream with $M_\infty = 8$. Harten's entropy correction[7] was used to eliminate the expansion shock that would otherwise form as the flow expands around the cylinder: For $|\lambda_i| < 2\epsilon$,

$$\hat{\lambda}_i^{\pm} = \pm\left(\frac{\lambda_i^2}{4\epsilon} + \epsilon\right)$$

with $\epsilon = 0.01$. Dissipation was also added on the line of symmetry by applying Harten's device to the linear field. A mesh with 35 normal and 26 circumferential points was used. The computed pressure distribution on the surface of the cylinder is compared to the solution of Lyubimov and Rusanov,[8] and to Roe's scheme (using the same amount of added dissipation) in Fig. 2.

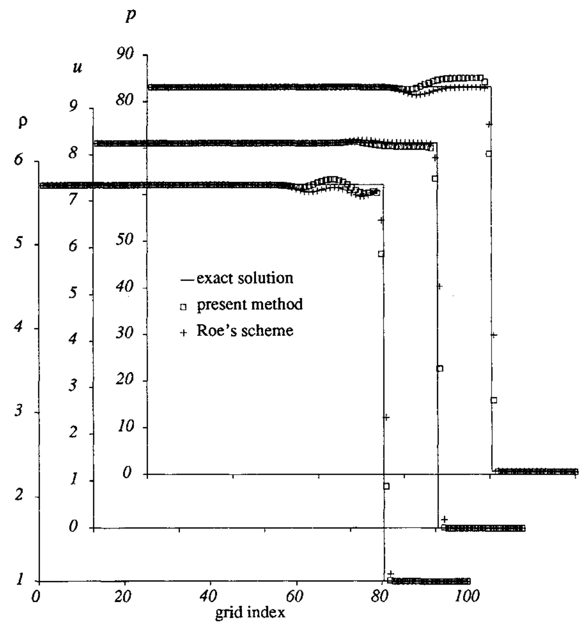

Fig. 1 Moving shock problem.

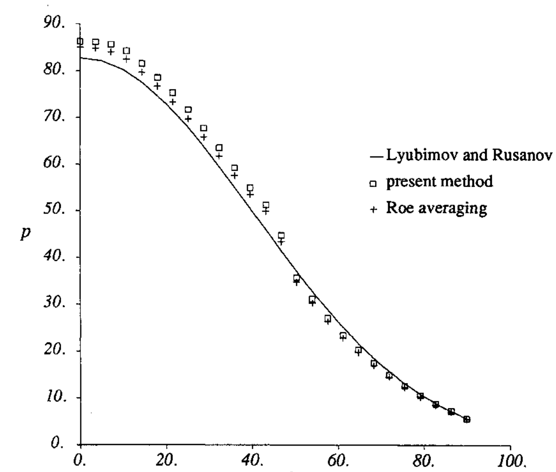

Fig. 2 Pressure distribution on the cylinder.

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